Self-organized criticality in a sheared granular stick-slip system

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We present an analysis of results obtained from a mechanical apparatus consisting of an annular plate shearing over a granular bed. The size, energy dissipation, and duration of slips in the system exhibit power-law distributions and a $1/f^2$ power spectrum, in accordance with self-organized criticality. We draw similarities with earthquakes.

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Bak, Tang, and Wiesenfeld introduced the concept of self-organized criticality (SOC) in the 1980s [1], suggesting that the theory was the underlying mechanism behind the widespread scale invariance observed in diverse natural systems. Computational models suggest that SOC is the underlying mechanism behind systems such as forest fires [2], earthquakes [3,4], economics [5], and evolution [6], among others. Jensen [7] and Bak [8] provide reviews.

Surprisingly, there are few laboratory experiments that exhibit SOC. Frette *et al.* [9] examined the occurrence of avalanches in a one-dimensional rice pile. The results appear consistent with SOC, an improvement over similar experiments with sand [10], where only small avalanches in a sandpile are power-law-dependent. Other experiments proposed to exhibit SOC include stick-slip motion in the form of weighted sandpaper pulled across a carpet [11], droplet formation [12], and avalanche effects in superconductors [13]. However, the short range over which scale-invariant behavior is often observed raises questions as to whether SOC is truly manifest in these experiments.

Another method available for studying the avalanche-type effects intrinsic to SOC, which, to our knowledge, has not previously been explored for this purpose, is to externally shear a confined granular medium in a Couette geometry. A number of recent notable studies have been conducted on the physics of sheared granular media, including investigations of friction [14], phase transitions [15,16], stress fluctuations [17,18], jamming states [19], and dragging [20]. While the physical cause of stress fluctuations has been explored, no statistical analyses of these fluctuations have been made. Observations of the power spectra of these fluctuations typically reveal $1/f^2$ signatures. Such a signature is consistent with SOC [21]; however, it may also be the result of a simple sawtooth signal, or the high-frequency end of a Poisson process [7]. To identify a critical state, temporal *and* spatial quantities must exhibit power-law dependence [7].

In this paper, we examine the statistics of stick-slip motion in a slowly sheared granular medium, including the size, energy, and duration of events. We demonstrate that the statistics are consistent with scale-invariant behavior over several decades, and suggest that their origin lies within a self-organized critical process.

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Our apparatus, shown in Fig. 1, consists of an annular plate that is driven over the surface of a granular medium (tapioca particles of diameter $5 \leqslant d \leqslant 2000$ μm) in a stick-slip fashion. The granular medium is confined to a circular channel of width 80 mm and mean diameter 280 mm. The plate is driven by the action of a motor via a torsion spring. In this way, the motor winds the torsion spring, increasing the torque (stress) on the plate. Ultimately, friction with the medium can no longer sustain the applied stress, the medium gives way, and the plate spins. A 5 mm gap is allowed at either edge of the plate to prevent contact with the channel walls. We measure the rotation of the drive axle above and below the torsion spring using rotary encoders, with a resolution of 1.3 arc min. The encoders are polled at 1 kHz and their states are stored for subsequent analysis. While this paper considers only the statistical behavior of the system over large periods of time, scan rates up to 10 kHz will permit the future study of individual events.

The torsion spring is manufactured from 3-mm-diam steel wire to BS5216 (III). This yields a deformation torque of

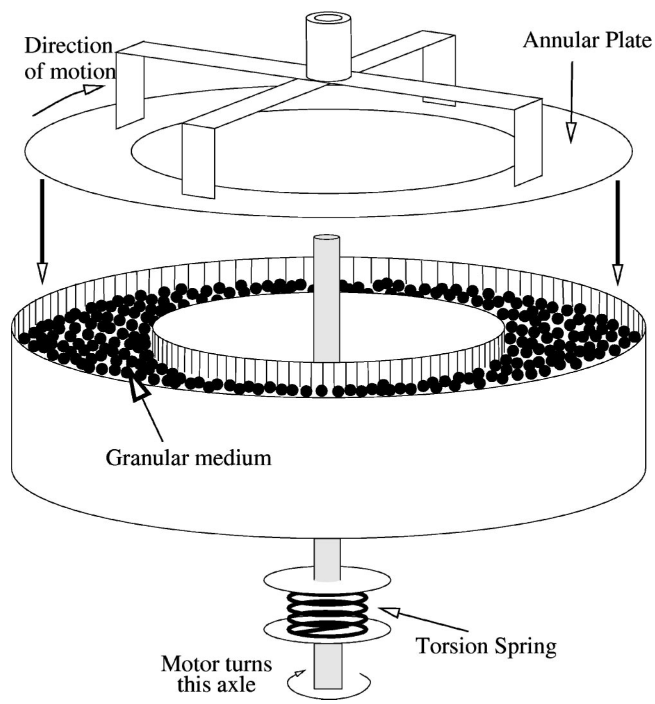

FIG. 1. Schematic of the apparatus. An annular plate rotates over a granular medium in a stick-slip fashion.

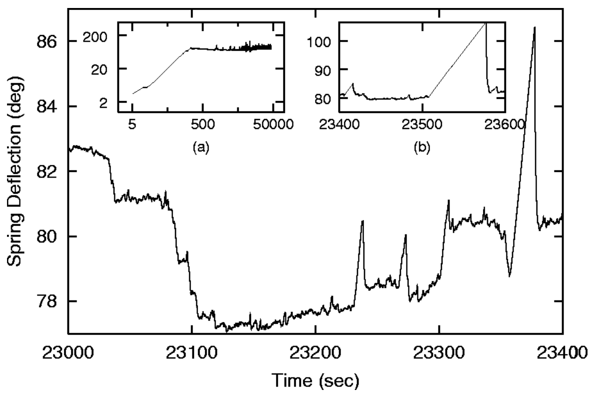

FIG. 2. The deflection of the torsion spring as a function of time. Fluctuations at all scales are seen. Inset (a) shows the initial wind-up to the stationary state on log-log axes, and inset (b) shows the slow wind-up to a rapid large event.

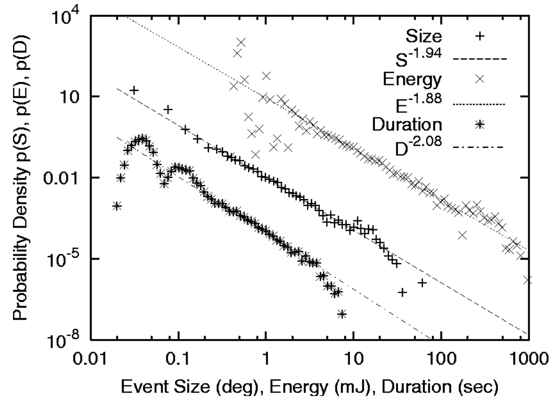

FIG. 3. The probability density distribution of event sizes, energies, and durations. The best-fit lines are shown with slopes $\lambda=1.94$, $\alpha=1.88$, $\tau=2.08$. The data are scaled vertically for clarity.

approximately 3250 N mm. The torsion spring used has nine coils, giving a maximum deflection of 180° and a rate of 18 N mm/deg. Using results from preliminary experiments (and several broken torsion springs), we estimate that the medium can subtend a maximum torque of approximately 2300 N mm, well within the spring's elastic limit.

Paper walls are attached to the plate to prevent particles from climbing on top, and a layer of tapioca is glued onto the plate to ensure a granular shearing plane. The torsion spring can be wound at a rate of $(0.4° \leqslant \omega \leqslant 32°)\mathrm{sec}^{-1}$. The largest tapioca particles subtend approximately 0.8° to the central axle. Before each experiment, we reset the medium by combing it to its maximum depth (approximately 30 mm). The surface is then leveled and the plate is placed on top. This initialization process will compress the medium to a volume fraction greater than that of the steady state. As the stress builds up, the force exerted by the central axle on the plate is sufficient to hold the plate at a constant height, providing a constant volume for the medium. However, the medium may partially dilate through the gap between plate and channel walls, though such dilation would be quite limited. In this paper, we present the results of an experiment where the torsion spring is driven at 0.4° sec^{-1}. The system is similar to that of Miller *et al.* [17], though our plate is driven via a torsion spring, and the plate is not free to move vertically.

That the system evolves to a statistically stationary state is well demonstrated by Fig. 2, inset (a), showing the evolution of the deflection of the torsion spring for the entire experiment. It is clear that the deflection, initially zero, rises to 80° and fluctuates about that value for the remainder of the experiment.

Figure 2 shows the deflection of the torsion spring as it varies in time. The most striking behavior observed is stick-slip motion. As seen, events can occur spontaneously, or immediately following a build-up period. The normal behavior of the device consists of such events over all scales. Figure 2, inset (b) shows the occurrence of a large event. Here we see that the torsion spring slowly winds up until the plate is no longer stable and undergoes slip, thus rapidly reducing the deflection.

Note that, for approximately 54% of its motion, the device is observed to exhibit steady sliding at the driving velocity. This occurs when the constant force from the motor is just sufficient to keep the medium fluidized, without being sufficient to accelerate the plate. We do not identify this continual sliding with the more intermittent behavior of slip events, and it is eliminated from the analyses below. Thompson and Grest [15] observe intermittent periods of uniform sliding within the stick-slip motion of their similar system.

The statistical analyses involve computing the probability density distribution of three event parameters: size S (°), computed as the angle through which the top plate moves, energy dissipation E (mJ), computed as the amount of energy the torsion spring loses by unwinding, and duration D (sec), as shown in Fig. 3. Here we see that an event has size S with probability $p(S)\sim S^{-\lambda}$, $\lambda=1.94\pm0.03$, dissipates energy E with power-law exponent $\alpha=1.88\pm0.04$, and has duration D with power-law exponent $\tau=2.08\pm0.04$. The fluctuations observed on these graphs at small values of the ordinate are due to the discrete nature of the measurement system.

In Fig. 4, the power spectrum of the deflection of various torsion springs is shown. For the experiment discussed in this paper (middle curve), the power spectrum is best fitted by a power law of slope $\beta=2.02\pm0.03$ over five decades.

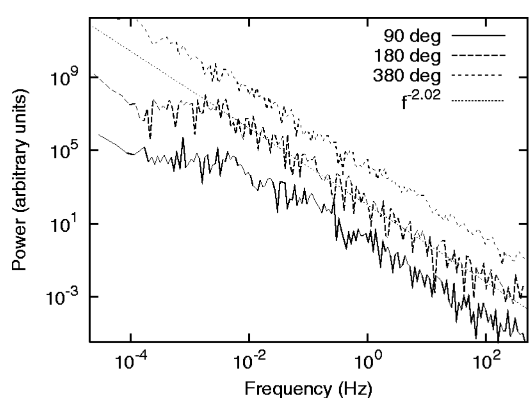

FIG. 4. The power spectrum of the deflection of the torsion spring, for torsion springs of various maximum deflection. The dotted line is the best fit to the 180° spring, with slope $\beta=2.02\pm0.03$.

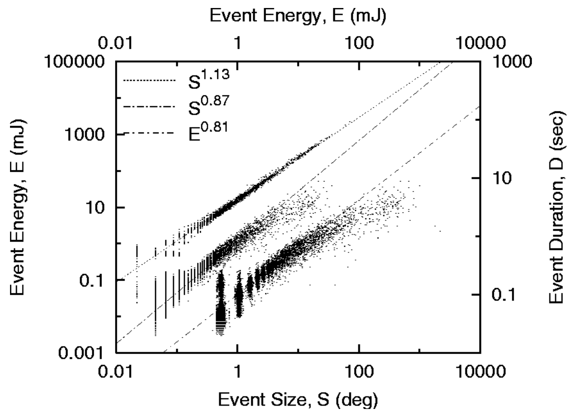

FIG. 5. Scaling relations of event parameters. The top curve is energy vs size, $\gamma_3 = 1.13$; the middle curve is duration vs size, $\gamma_2 = 0.87$; and the bottom curve is duration vs energy, $\gamma_1 = 0.81$.

There is a low-frequency cutoff point at approximately 2 mHz, below which white noise dominates. The low-frequency cutoff is related to the details of the driving used, and replacing the torsion spring by one with a lower or higher maximum deflection (90° or 380°) moves the cutoff to a higher or lower frequency, respectively. Indeed, for the experiment with the 380° spring, the cutoff frequency has an associated period that is longer than the duration of the experiment (4.75 h), and hence is not revealed by the analysis.

After Jensen [7], we assume that the size, energy, and duration of each event scale to one another: $\langle D(E)\rangle \sim E^{\gamma_1}$, $\langle D(S)\rangle \sim S^{\gamma_2}$, and $\langle E(S)\rangle \sim S^{\gamma_3}$, with $\gamma_1 = (\alpha - 1)/(\tau - 1)$, $\gamma_2 = (\lambda - 1)/(\tau - 1)$, and $\gamma_3 = (\lambda - 1)/(\alpha - 1)$ with $\gamma_1\gamma_3/\gamma_2 = 1$. Thus we compute $\gamma_1 = 0.81 \pm 0.06$, $\gamma_2 = 0.87 \pm 0.05$, and $\gamma_3 = 1.07 \pm 0.06$. These scaling relations are compared to the data in Fig. 5. There is good agreement except for the energy to size scaling, where $\gamma_3 = 1.13$ (shown) is a better fit. This value, however, is within the error bars computed.

Events are found to exhibit scale-invariant power-law statistics over two to three decades (Fig. 3), indicative of an underlying fractal structure, and characteristic of a critical state. The critical nature of the system is also apparent from the scaling relations of these observables, shown in Fig. 5. As the stationary state arises under nonspecific conditions (i.e., the annular plate is simply rested on the granular bed and the system is driven arbitrarily ''slowly''), we assume that SOC is the mechanism at work. In general, our experiments show that SOC is robust and arises for a wide range of conditions, including various granular media, driving rates, and media preparation, though the exponents may vary [22].

The results presented here compare well with those of previous SOC experiments. Frette *et al.* [9] found that avalanches in a rice pile dissipate energy with a power-law probability of exponent 2.02. Manna and Khakhar [23] found that avalanches within the bulk of a granular medium occur with a power-law exponent 1.7. The OFC [3] model assumes a power-law exponent of $0.22 \leqslant B \leqslant 2.5$ for event energy, depending on the level of conservation, easily encapsulating our result.

The total stress in the system, measured as the deflection of the torsion spring, fluctuates with $1/f^2$ noise. As we can reduce the low-frequency cutoff, presumably to an arbitrarily low frequency, this suggests that the system has a diverging correlation time. The water-droplet experiment [12], the BTW model [1], Feder and Feder's experiment [11], and the rice-pile experiment [9], all proposed as examples of SOC, also exhibit $1/f^2$ noise.

We now explore how the critical state present in our system mechanically arises within the granular medium. In the area of soil mechanics, it is well established that a soil under continual shear strain will dilate or compress to a volume fraction, at which further strain can occur, without any change in applied shear stress or volume fraction [24]. Our sheared granular medium also moves towards such a constant stress state [Fig. 2, inset (a)].

The volume fraction, ν, at this attractor has been denoted ''critical'' by soil mechanists [24], but by this, they do not imply criticality in the thermodynamic sense. Aharonov *et al.* [16], in a model of a sheared granular medium, investigated the occurrence of this ''critical'' state. Under static conditions free of gravity, a rigidity phase transition was found in which the coordination number Z (a measure of the average number of contacts per grain) exhibited a first-order transition at the ''critical'' volume fraction ν_c, while the shear modulus, G, of the medium followed a second-order phase transition with $[G \sim (Z - Z_c) \sim (\nu - \nu_c)^{\alpha}]$. Makse *et al.* [25] obtained similar results for a compressed granular material. We believe that this second-order transition is intimately related to the origin of the SOC state observed by us. In addition, the minimum coordination number will be nonzero in a gravitational system, and thus the coordination number may, under gravity, also obey a continuous phase transition.

The configuration of the sheared granular bed of the experiments conducted here resembles the constant volume system of Aharonov *et al.* [16], which exhibits stick-slip behavior when compressed at or above the ''critical'' volume fraction. Howell *et al.* [18] also note that, as the critical volume fraction is approached, the intermittence of stress fluctuations increases while the mean velocity of the sheared disks decreases. This also seems consistent with slip-stick motion. Though the volume fraction is externally controlled in these experiments, our system can self-organize itself to this point in a manner similar to the constant force (i.e., variable volume) system of Aharonov *et al.* [16]. Such self-organization may be accomplished by two methods, both of which assume that the initial volume fraction is higher than the critical value. First, the medium may dilate, being driven by sideways displacement of the granular medium into the gaps between the shearing plate and the channel walls. Alternatively, or together with the above process, we envisage an initial period of readjustment under gravity, which compacts the underlying bed leaving a shearing region of reduced volume fraction. We expect a shear gradient within the medium [15], with variations in the volume fraction occurring both spatially and temporally between events. On average, however, it is expected that the volume fraction within the shearing region will remain close to its critical value.

In a stick phase, it is expected that the stress within the medium is carried by chains of particles [19,25]. It is interesting that the length of such stress chains is seen to grow as the critical volume fraction is approached [18]. Makse *et al.* [25] have also noted the lengthening of relaxation times to equilibrate their static system near this critical volume fraction. If this behavior was indicative of diverging length and time scales, it would be consistent with the observed fractal dynamics of our system, and hence its criticality.

Finally, we note the obvious similarity between the sheared granular system presented here and that of an earthquake fault. It is particularly interesting that slip events within this system dissipate energy with power-law probability in excellent agreement with the Gutenburg-Richter power law. In the latter, earthquakes dissipate energy E with power-law probability $p(E)\sim E^{-b-1}$, $0.8\leqslant b\leqslant 1.05$ [3], where we obtain an exponent $b=0.88\pm 0.04$. A preliminary investigation of the relationship between earthquakes and our system is reported elsewhere [26].

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